Hopping expansion as a tool for handling dual variables in lattice models

Christof Gattringer [a] *

[a]Massachusetts Institute of Technology, Center for Theoretical Physics
77 Massachusetts Avenue, Cambridge MA 02139 USA

The hopping expansion of 8-vertex models in their Grassmann representation is studied. We use the functional similarity of the Ising model in this expansion with the hopping expansion of 2-D Wilson fermions to show that the lattice fermions are equivalent to the Self-avoiding Loop Model at bending rigidity $1/\sqrt{2}$.

1. Grasssmann representation for 8-vertex models and the hopping expansion

Many lattice models have representations in terms of dual variables. E.g. the partition function of the 2-dimensional Ising model can be written as a sum over all possible Peierls contours with a weight given by $e^{-2\beta}$ to the power of the length of the contours. A generalized model which includes many dual models of lattice systems is the 8-vertex model [1,2]. In the 8-vertex model the dual variables (loops) are decomposed into their elements and each of those elements (vertices) is assigned a weight. The vertices can be viewed as 8 different tiles and are shown in Fig. 1. The figure also gives the weights ω_i in the notation of [1].

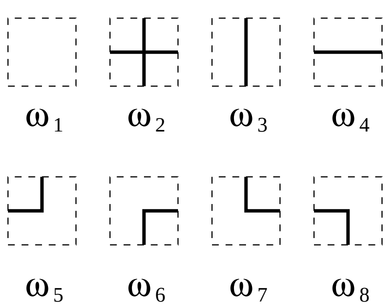

Figure 1. *Vertices and their weights in the 8-vertex model.*

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The loops are obtained as tilings of the plane such that there is never an open end for the fat line. The partition function of the 8-vertex model is then given by

$$Z_{8v} \;=\; \sum_{loops} \prod_{i=1}^{8} (\omega_i)^{n_i} \;, \tag{1}$$

where ω_i is the weight of the i-th vertex, and n_i gives the abundance of this vertex in the loop configuration.

It is known [3,4], that the 8-vertex model has a representation as an integral over Grassmann variables

$$Z_{8v} \;=\; \int [d\eta]\; e^{-S[\eta]} \;. \tag{2}$$

Here to each site x of the dual lattice a vector of 4 Grassmann variables

$$\eta(x) \;=\; (\eta_{+1}(x), \eta_{-1}(x), \eta_{+2}(x), \eta_{-2}(x))^T \;, \tag{3}$$

was assigned. The action S is a sum of propagator-, corner- and monomer-terms ($S = S_p + S_c + S_m$), with

$$\begin{aligned}
S_p &= \sum_x \big[\; a\;\eta_{+1}(x)\eta_{-1}(x+\hat{1}) \\
&\qquad + b\;\eta_{+2}(x)\eta_{-2}(x+\hat{2})\big] \;, \\
S_c &= \sum_x \big[\; c\;\eta_{+1}(x)\eta_{-2}(x) + d\;\eta_{+2}(x)\eta_{-1}(x) \\
&\qquad + e\;\eta_{-2}(x)\eta_{-1}(x) + f\;\eta_{+2}(x)\eta_{+1}(x)\big] \;, \\
S_m &= \sum_x \big[\; g\;\eta_{-1}(x)\eta_{+1}(x) + h\;\eta_{-2}(x)\eta_{+2}(x)\big] .
\end{aligned} \tag{4}$$

When expressing the weights ω_i through the coefficients $a, b, \ldots h$ as

$\omega_1 = -cd - ef + gh \;,\; \omega_2 = -ab \;,\; \omega_3 = bg \;,$

$$\omega_4 = ah \ , \ \omega_5 = c\sqrt{ab} \ , \ \omega_6 = d\sqrt{ab} \ ,$$
$$\omega_7 = e\sqrt{ab} \ , \ \omega_8 = f\sqrt{ab} \ , \tag{5}$$

it can be shown by explicit expansion that the Grassmann integral (2) reproduces the partition function (1). Since (4) can be diagonalized by Fourier transform, finding a Grassmann representation (2), (4) corresponds to solving the model. This is reflected by the fact, that the choice (5) obeys the free fermion condition $\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8$, which is a sufficient condition for an analytic solution of the 8-vertex model [1].

Anti-symmetrization turns the action into a quadratic form $S = \frac{1}{2}\sum_{x,y} \eta(x)^T K(x,y)\eta(y)$ with kernel

$$K \ = \ M \ + \ \sum_{\mu=\pm 1}^{\pm 2} P_\mu \ , \tag{6}$$

where M, P_μ have lattice indices x, y and indices i, j acting on the Grassmann 4-vectors (3). These matrices are given by

$$M(x,y) \ = \ \delta_{x,y} \left(\begin{array}{cccc} 0 & -g & -f & +c \\ +g & 0 & -d & -e \\ +f & +d & 0 & -h \\ -c & +e & +h & 0 \end{array} \right) \ , \tag{7}$$

and

$$(P_{+1}(x,y))_{i,j} \ = \ a \ \delta_{x+\hat{1},y}\delta_{i,1}\delta_{j,2} \ ,$$
$$(P_{+2}(x,y))_{i,j} \ = \ b \ \delta_{x+\hat{2},y}\delta_{i,3}\delta_{j,4} \ ,$$
$$P_{-1} = -P_{+1}^T \quad , \quad P_{-2} = -P_{+2}^T \ . \tag{8}$$

Besides the termwise expansion which gives the partition function (1), the Grassmann integral (2) can also be evaluated as a Pfaffian giving

$$Z_{8v} \ = \ \mathrm{Pf}\, K \ = \ \sqrt{\det K} \ . \tag{9}$$

In the second equality we used the fact that K is anti-symmetric and thus its Pfaffian reduces to the root of a determinant. The matrix M has determinant $(gh - ef - cd)^2$ which after inserting (5) and using the free fermion condition reduces to $\det M = \omega_1^2$. Thus for 8-vertex models with $\omega_1 \neq 0$, the matrix M can be inverted and one can transform to

$$Z_{8v} = \sqrt{\det M}\sqrt{\det[1+H]} \ = \ \exp\left(-\frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ \mathrm{Tr} H^n \right) \ , \tag{10}$$

where we used the well known formula for the determinant as the exponential of the trace of the logarithm and defined the hopping matrix H as

$$H \ \equiv \ \sum_{\mu=\pm 1}^{\pm 2} M^{-1} P_\mu \ . \tag{11}$$

The series in the exponent converges for values of the parameters where $\parallel H \parallel < 1$. Since the matrices P_μ contain the hopping factors $\delta_{x+\mu,y}$ only terms that correspond to closed loops can contribute to (10). These closed loops are however related to the loops used in the original definition (1) of the model and thus the exponent in (10) is in principle the free energy of the model expressed in terms of loops. The only remaining problem is to explicitly solve the traces over powers of the hopping matrix H in (10). For the general case this will be discussed elsewhere [5].

The Ising model can be obtained from the 8-vertex model by setting

$$\omega_1 = 1 \ , \ \omega_2 = e^{-4\beta} \ , \ \omega_j = e^{-2\beta} \ , \ j \ = 3, ...8. \tag{12}$$

For this case the hopping expansion (10) gives [6]

$$Z_i = \exp\left(\sum_l \frac{(-1)^{n(l)}}{I(l)} e^{-2\beta|l|} \right) \ . \tag{13}$$

Here l runs over all closed, simply connected loops on the lattice, $n(l)$ denotes the number of self-intersections, $I(l)$ gives the number of times the loop l iterates its entire trace and $|l|$ is the length of the loop.

2. Equivalence of 2-D Wilson fermions with the Self-avoiding Loop Model at bending rigidity $1/\sqrt{2}$

In this section we give an example how representations of the type (10) can be used. We will show that 2-dimensional Wilson fermions on the lattice are equivalent to the Self-avoiding Loop Model at bending rigidity $1/\sqrt{2}$. For an alternative derivation of this result see Scharnhorst's paper [7].

The action for 2-dimensional Wilson fermions [8] is given by

$$S \;=\; \overline{\psi} M \psi \quad , \quad M \;=\; 1 \;-\; \kappa Q \; , \tag{14}$$

with

$$Q(x,y) \;=\; \sum_{\mu=\pm 1}^{\pm 2} \Gamma_\mu \; \delta_{x+\mu,y} \; , \tag{15}$$

where $\Gamma_{\pm\mu} = \frac{1}{2}[1 \mp \sigma_\mu], \;\mu = 1,2$. Here σ_1, σ_2 denote the Pauli matrices and the hopping parameter κ is related to the bare fermion mass m via $\kappa = (m+2)^{-1}$. Since the action for the lattice fermions is a bilinear form the partition function is not a Pfaffian but a determinant

$$Z_w(\kappa) \;=\; \int [d\overline{\psi}][d\psi] \; e^{-S} \;=\; \det M \;=$$

$$\det[1-\kappa Q] \;=\; \exp\left(- \sum_{n=1}^{\infty} \frac{\kappa^n}{n} \mathrm{Tr} Q^n \right) \; . \tag{16}$$

Again the contributions to the traces in the exponent correspond to closed loops, and the remaining problem is to compute the traces of the matrices Γ_μ ordered along the links of the loops. This problem was solved in [9] by realizing that the Pauli matrices give rise to a representation of rotations on the lattice. The resulting hopping expansion is

$$Z_w \;=\; \exp\left(2 \sum_l \kappa^{|l|} \frac{(-1)^{n(l)}}{I(l)} \left(\frac{1}{\sqrt{2}}\right)^{c(l)} \right) \; . \tag{17}$$

This result is very similar to Formula (13) for the Ising model, with the only difference of an extra weight $1/\sqrt{2}$ for each corner of a loop l (here $c(l)$ denotes the number of corners in a loop l).

When expanding the exponential functions in (13) or (17) one obtains contributions from products of loops l appearing in the exponent, such that in these products some links are multiply occupied. In addition already single loops l can retrace some of their links such that they are multiply occupied. For the Ising model we know that all the contributions with multiply occupied links cancel since the Peierls contour representation discussed in the beginning allows only for simply occupied links (see [6] for a detailed discussion). This cancellation mechanism is independent of corner weights since two loop configurations which differ in their number of corners necessarily have different occupancy numbers at some links and thus only contributions with equal numbers of corners can cancel each other. Hence, using the similarity between (17) and (13) we can conclude that also in an expansion of the exponential in (17), terms with multiply occupied links cancel. The remaining terms inherit the corner factors $1/\sqrt{2}$, and the self-intersection factor $(-1)^{n(l)}$ in (17) leads to a cancellation of terms with self-intersections (i.e. the corresponding vertex model has weight $\omega_2 = 0$ and thus the loops are self-avoiding). Thus we end up with (see [5] for the detailed proof)

$$\sqrt{Z_w(\kappa)} \;=\; Z_{salm}\left(\kappa, \frac{1}{\sqrt{2}}\right) \; , \tag{18}$$

with the Self-avoiding Loop model defined as

$$Z_{salm}(z,\eta) \;=\; \sum_\gamma z^{|\gamma|} \, \eta^{c(\gamma)} \; . \tag{19}$$

Here γ runs over all self-avoiding contours, $c(\gamma)$ gives the number of corners, η is the bending rigidity, z is the monomer weight and $|\gamma|$ denotes the length of the contour.

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